Unique Continuation and Rigidity for the Einstein Vacuum Equations: A Formulation Study

Joseph Scott Penman

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Abstract

This is a formulation study exploring whether the endpoint unique continuation framework (form-bounded potentials + Carleman estimates + doubling + frequency rigidity) is structurally compatible with the Einstein vacuum equations. Unlike previous systems (Navier–Stokes, Yang–Mills, wave maps), Einstein is fundamentally different: the operator depends on the solution, light cones move, Carleman geometry is dynamical, and unique continuation can fail on spacetimes with trapped null geodesics. This document asks: **Is the UC-form-frequency pipeline even definable here?** The goal is not to claim closure, but to identify the exact geometric obstructions to UC-rigidity in general relativity.

Contents

1 Introduction: A Formulation Study

1.1 Purpose of This Document

This document is a **formulation study**, not a proof. Its purpose is to explore whether the endpoint unique continuation framework (form-bounded potentials + Carleman estimates + doubling + frequency rigidity) is structurally compatible with the Einstein vacuum equations.

Unlike previous systems in this program:

- **Navier–Stokes:** Semilinear parabolic, fixed light cones
- **Yang–Mills (parabolic/hyperbolic):** Quasilinear with known dispersion, fixed or slowly varying cones
- **Wave maps:** Geometric evolution with fixed Minkowski background

Einstein is fundamentally different:

- The operator $\Box_g$ depends on the solution g itself
- Light cones move with the metric
- Carleman weight geometry is dynamical
- Unique continuation can be **false** on spacetimes with trapped null geodesics

1.2 The Central Question

Is the UC-form-frequency pipeline even definable here?

This document asks:

1. Can we fix a gauge (harmonic/wave gauge) and linearize?
2. Can we define a form functional for the curvature Rm?
3. Can we define backward light cones in curved spacetime?
4. Can we state (not prove) a conjectural Carleman inequality?
5. What are the exact geometric obstructions?

1.3 Expected Outcome

The likely outcome is **not closure**, but:

- Identification of exact geometric obstructions to UC-rigidity in GR
- Understanding of where the endpoint framework breaks
- A deep and honest result about the limits of quantitative unique continuation in general relativity

This is a reality check. If the framework fails here, that failure is informative and valuable.

1.4 Structure

- **Section 2:** Gauge fixing and linearization
- **Section 3:** Curvature as a form-bounded potential (candidate definition)
- **Section 4:** Backward light cones in curved spacetime
- **Section 5:** Carleman feasibility (conjectural, not proven)
- **Section 6:** Obstructions (lack of coercivity, trapped null geodesics, gauge degeneracy, failure of backward uniqueness)

1.5 Conventions

We work in 3+1 dimensions unless otherwise stated. The metric signature is $(-, +, +, +)$. All statements are **formulations** or **questions**, not proven theorems unless explicitly marked as such.

2 Gauge Fixing and Linearization

2.1 The Einstein Vacuum Equations

The Einstein vacuum equations are:

$$R_{\mu\nu} = 0,$$

where $R_{\mu\nu}$ is the Ricci tensor of the metric $g_{\mu\nu}$.

In coordinates, this is a system of 10 nonlinear PDEs for the 10 components of the metric tensor.

2.2 Harmonic/Wave Gauge

To linearize, we must fix a gauge. The standard choice is the **harmonic gauge** (also called **wave gauge**):

$$\Box_g x^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0,$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols.

In harmonic gauge, the Einstein equations become:

$$\Box_g g_{\mu\nu} + Q_{\mu\nu}(g, \partial g) = 0,$$

where $Q_{\mu\nu}$ is a quadratic form in the first derivatives of g.

2.3 Linearization

Let $g_{\mu\nu}$ be a solution to the Einstein vacuum equations, and let $h_{\mu\nu}$ be a perturbation. The linearized Einstein equations in harmonic gauge are:

$$\Box_g h_{\mu\nu} + 2R_{\mu\alpha\nu\beta}h^{\alpha\beta} = 0,$$

where:

- $\Box_g = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the covariant d'Alembertian
- $R_{\mu\alpha\nu\beta}$ is the Riemann curvature tensor of the background metric g
- $h^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}h_{\mu\nu}$ is the raised perturbation

2.4 UC Operator Form (Candidate)

Candidate UC Operator Form

The linearized Einstein equations take the form:

$$\Box_g h_{\mu\nu} + V_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} = 0,$$

where:

- $\Box_g$ is the covariant wave operator (depends on g)
- $V_{\mu\nu}^{\alpha\beta} = 2R_{\mu\alpha\nu\beta}$ is the curvature potential (depends on g)

Key difference from previous systems: Both the operator and the potential depend on the solution g itself. This is not a fixed operator with a form-bounded potential; it is a **quasilinear system** where the linearization depends on the background.

2.5 Questions

Question 2.1 (Gauge Stability)**.** Is the harmonic gauge condition preserved under the linearized evolution? If not, what is the gauge drift term, and can it be controlled?

Question 2.2 (Background Regularity)**.** What regularity do we need on the background metric g to make sense of $\Box_g$ and $R_{\mu\alpha\nu\beta}$? Is this regularity preserved under the Einstein evolution?

Question 2.3 (Operator Coercivity)**.** Is $\Box_g$ coercive on tensor fields? In Minkowski space, $\Box$ is coercive. In curved spacetime, does the curvature break coercivity?

3 Curvature as a Form-Bounded Potential

3.1 Candidate Form Functional

In previous systems (Navier–Stokes, Yang–Mills, wave maps), we defined a form functional:

$$\mathcal{A}_{|V|,r}(t) = \sup_{g \in H_0^1(B_r(t)) \setminus \{0\}} \frac{\int_{B_r(t)} |V||g|^2}{\int_{B_r(t)} |\nabla g|^2}.$$

For Einstein, the potential is the curvature tensor $R_{\mu\alpha\nu\beta}$. A candidate form functional would be:

$$\mathcal{A}_{|Rm|,r}(t) = \sup_{h \in H_0^1(B_r(t)) \setminus \{0\}} \frac{\int_{B_r(t)} |R_{\mu\alpha\nu\beta}||h^{\alpha\beta}|^2}{\int_{B_r(t)} |\nabla h|^2},$$

where h is a tensor field and $|R_{\mu\alpha\nu\beta}|$ is some norm of the curvature tensor.

3.2 Scale-Invariance Question

Question 3.1 (Scale-Invariance)**.** Is $R_{\mu\alpha\nu\beta}$ even in the right scale-invariant class for form control?

In Minkowski space, $R = 0$. For a perturbation, the curvature scales like $|x|^{-2}$ in the critical case. But in curved spacetime, the background curvature itself may not be scale-invariant.

What is the natural scale-invariant norm for $|R_{\mu\alpha\nu\beta}|$? Is it $L^{3/2,1}$ (as in Navier–Stokes), or something else?

3.3 Energy Control (Conjectural)

In previous systems, we proved:

$$\text{Energy} \Rightarrow \text{Form-boundedness.}$$

For Einstein, the energy is:

$$E(h,t) = \int_{\Sigma_t} \left(|\partial_t h|^2 + |\nabla h|^2 + |h|^2 \right) d\mu_g,$$

where Σ_t is a spacelike hypersurface and $d\mu_g$ is the volume form.

Question 3.2 (Energy to Form Control)**.** Can we prove (or is it even true) that:

$$E(h,t) < \infty \Rightarrow \mathcal{A}_{|Rm|,r}(t) < \infty?$$

Or does the curvature $R_{\mu\alpha\nu\beta}$ itself need to be controlled independently?

3.4 Time Integrability Question

In previous systems, we had:

$$r \in L_t^2((-r,0)).$$

Question 3.3 (Time Integrability)**.** What replaces L_t^2 in curved spacetime?

In curved spacetime, time is not a global coordinate. We work on a foliation $\{\Sigma_t\}$ of spacelike hypersurfaces. What is the natural time-integrability condition?

Is it:

$$\int_{-r}^{0} \mathcal{A}_{|Rm|,r}(t)^2 dt < \infty?$$

Or do we need a different measure (e.g., proper time along geodesics)?

3.5 Good-Time Selection (If Definable)

In previous systems, we defined good-time sets via Chebyshev's inequality:

$$I_{r,\text{good}}(t_0) = \{t \in (-r,0) : K_{\text{uc}}(r,t) \le \Lambda_r\}.$$

Question 3.4 (Good-Time Selection)**.** Can we define good-time sets in curved spacetime? What is the time coordinate? Do we use:

- Coordinate time (if a global time function exists)?
- Proper time along a timelike geodesic?
- Something else?

4 Backward Light Cones in Curved Spacetime

4.1 The Fundamental Difference

In Minkowski space, backward light cones are well-defined:

$$C^-(x_0, t_0) = \{(t, x) : t < t_0, |x - x_0| < t_0 - t\}.$$

In curved spacetime, light cones are **dynamical** and depend on the metric g itself.

4.2 Definition of Backward Light Cones

For a point $p \in \mathcal{M}$ in a spacetime $(\mathcal{M}, g)$, the **backward light cone** $C^-(p)$ is the set of all points q such that there exists a future-directed null geodesic from q to p.

More precisely:

$$C^-(p) = \{q \in \mathcal{M} : q \in J^-(p) \setminus I^-(p)\},$$

where $J^-(p)$ is the causal past of p and $I^-(p)$ is the chronological past of p.

4.3 Truncated Cones

For a radius $r > 0$ (measured in proper time or some other parameter), we can define a **truncated backward light cone**:

$$C_r^-(p) = \{q \in C^-(p) : \text{dist}(q, p) < r\},$$

where $\text{dist}(q, p)$ is the proper time along a null geodesic from q to p (if it exists).

Question 4.1 (Cone Regularity)**.** Are backward light cones in curved spacetime smooth? Do they have a well-defined boundary? What regularity do we need on g to ensure this?

4.4 Energy on Cones

In Minkowski space, we defined:

$$M_E(t, r) = \int_{\{t\}\times B_{-t}(0)} \left(|\partial_t h|^2 + |\nabla h|^2\right) dx.$$

In curved spacetime, we need to define energy on a spacelike slice of the backward light cone.

Question 4.2 (Energy Definition)**.** How do we define energy on a slice of a backward light cone in curved spacetime?

Options:

- Use a foliation $\{\Sigma_t\}$ of spacelike hypersurfaces and integrate over $\Sigma_t \cap C_r^-(p)$
- Use the stress-energy tensor $T_{\mu\nu}$ and integrate over a null hypersurface
- Something else?

4.5 Trapped Null Geodesics

Obstruction 4.3 (Trapped Null Geodesics). On spacetimes with trapped null geodesics (e.g., Schwarzschild black hole), backward light cones can be **non-compact** or **ill-defined**.

In such spacetimes, unique continuation can be **false**. This is a fundamental obstruction to the UC framework.

Question 4.4 (Trapping)**.** Can we restrict to spacetimes without trapped null geodesics? Is this a reasonable assumption, or does it exclude physically interesting cases (e.g., black holes)?

4.6 Cone Geometry and Carleman Weights

In previous systems, we used Carleman weights of the form:

$$\Phi(t, x) = |x|^2 - \beta t^2.$$

In curved spacetime, the geometry is dynamical.

Question 4.5 (Carleman Weight Geometry)**.** Can we define a Carleman weight Φ_g that:

- Is adapted to the backward light cone $C_r^-(p)$
- Has the right convexity properties for a Carleman estimate
- Depends on the metric g in a controlled way

Or does the dynamical geometry break the Carleman weight construction?

5 Carleman Feasibility (Conjectural)

5.1 Standard Carleman Estimate (Minkowski)

In Minkowski space, we have:

$$\tau\|\nabla h\|^2_{L^2(C_r^-)} + \tau^3\|h\|^2_{L^2(C_r^-)} \leq C\|e^{\tau\Phi}\Box(e^{-\tau\Phi}h)\|^2_{L^2(C_r^-)},$$

where $\Phi(t, x) = |x|^2 - \beta t^2$ is a Carleman weight.

5.2 Conjectural Carleman Estimate (Curved)

For the linearized Einstein equations in curved spacetime, a conjectural Carleman estimate would be:

$$\tau\|\nabla_g h\|^2_{L^2(C_r^-(p))} + \tau^3\|h\|^2_{L^2(C_r^-(p))} \leq C_g\|e^{\tau\Phi_g}\Box_g(e^{-\tau\Phi_g}h)\|^2_{L^2(C_r^-(p))},$$

where:

- $\Box_g$ is the covariant d'Alembertian
- Φ_g is a Carleman weight adapted to the curved geometry
- C_g depends on the metric g and the curvature

5.3 Questions

Question 5.1 (Carleman Estimate Existence). Does such a Carleman estimate exist in curved spacetime? What conditions on g are needed?

Known obstructions:

- **Lack of coercivity:** If $\Box_g$ is not coercive, the estimate may fail
- **Trapped null geodesics:** On spacetimes with trapping, the estimate may not hold
- **Dynamical geometry:** The weight Φ_g must be adapted to the moving light cones

Question 5.2 (Constant Dependence). How does the constant C_g depend on the metric g? Does it blow up as:

- The curvature becomes large?
- We approach a trapped null geodesic?
- The metric becomes degenerate?

5.4 Absorption of Curvature Potential

In previous systems, we absorbed the potential via form-boundedness:

$$\|e^{\tau\Phi}Vh\|_{L^2} \leq C\Lambda\|e^{\tau\Phi}\nabla h\|_{L^2}.$$

For Einstein, the potential is $V_{\mu\nu}^{\alpha\beta} = 2R_{\mu\alpha\nu\beta}$.

Question 5.3 (Potential Absorption). Can we absorb the curvature potential $R_{\mu\alpha\nu\beta}$ into the Carleman estimate? What form-boundedness condition do we need?

Is it:

$$\mathcal{A}_{|Rm|,r}(t) \leq \Lambda_r?$$

Or do we need a different condition (e.g., pointwise bounds on $|R_{\mu\alpha\nu\beta}|$)?

5.5 Three-Cone Inequality (Conjectural)

If a Carleman estimate exists, we could try to derive a three-cone inequality:

$$M_E(t, r_2) \leq CM_E(t, r_1)^{\alpha} M_E(t, r_3)^{1-\alpha} \exp(CK_{\mathrm{uc}}^{\mathrm{Einstein}}(r, t)),$$

where $M_E(t, r)$ is the energy on a slice of the backward light cone.

Question 5.4 (Three-Cone Feasibility). Is such a three-cone inequality even feasible in curved spacetime? What are the obstructions?

6 Geometric Obstructions to UC-Rigidity in GR

This section identifies the exact geometric obstructions that prevent the endpoint UC framework from working in general relativity.

6.1 Obstruction 1: Lack of Coercivity

Obstruction 6.1 (Lack of Coercivity). The covariant d'Alembertian $\Box_g$ may not be coercive on tensor fields in curved spacetime.

In Minkowski space, $\Box$ is coercive. In curved spacetime, the curvature can break coercivity, especially near:

- Singularities (e.g., Big Bang, black hole singularities)
- Regions of large curvature
- Degenerate metrics

Impact: If $\Box_g$ is not coercive, the Carleman estimate may fail, and the entire UC pipeline breaks down.

6.2 Obstruction 2: Trapped Null Geodesics

Obstruction 6.2 (Trapped Null Geodesics). On spacetimes with trapped null geodesics (e.g., Schwarzschild black hole), backward light cones can be non-compact or ill-defined.

In such spacetimes, unique continuation can be **false**. This is a fundamental obstruction.

Impact: The UC framework assumes that backward light cones are well-defined and compact. On trapped spacetimes, this assumption fails.

6.3 Obstruction 3: Gauge Degeneracy

Obstruction 6.3 (Gauge Degeneracy). The harmonic gauge condition $\Box_g x^\mu = 0$ may not be preserved under evolution, or may become degenerate.

Gauge drift terms can accumulate and break the linearization.

Impact: If the gauge is not stable, the linearized equation $\Box_g h_{\mu\nu} + 2R_{\mu\alpha\nu\beta}h^{\alpha\beta} = 0$ may not be valid, and the UC operator form breaks down.

6.4 Obstruction 4: Failure of Backward Uniqueness

Obstruction 6.4 (Failure of Backward Uniqueness). Even if a Carleman estimate exists, backward uniqueness can fail in general relativity.

Known counterexamples exist on spacetimes with:

- Closed timelike curves
- Naked singularities
- Exotic topologies

Impact: The entire blow-up exclusion argument relies on backward uniqueness. If this fails, the framework cannot close.

6.5 Obstruction 5: Dynamical Geometry

Obstruction 6.5 (Dynamical Geometry). The light cones move with the metric g. The Carleman weight geometry is dynamical.

Unlike previous systems where the geometry is fixed (Minkowski) or slowly varying (Yang–Mills, wave maps), in Einstein the geometry is fully dynamical.

Impact: The Carleman weight Φ_g must be adapted to the moving geometry. This may be impossible or may require conditions on g that are not preserved under evolution.

6.6 Obstruction 6: Operator Dependence on Solution

Obstruction 6.6 (Operator Dependence). The operator $\square_g$ and the potential $R_{\mu\alpha\nu\beta}$ both depend on the solution g itself.

This is fundamentally different from previous systems, where we had a fixed operator with a form-bounded potential.

Impact: The form-boundedness condition $\mathcal{A}_{|Rm|,r}(t) \leq \Lambda_r$ must be checked on the solution g itself. This creates a circularity: we need g to be regular to define the form functional, but we need the form functional to prove regularity.

6.7 Summary: Where the Framework Breaks

The endpoint UC framework breaks in general relativity due to:

1. **Lack of coercivity** of $\square_g$ in curved spacetime
2. **Trapped null geodesics** making backward light cones ill-defined
3. **Gauge degeneracy** breaking the linearization
4. **Failure of backward uniqueness** on exotic spacetimes
5. **Dynamical geometry** making Carleman weights ill-defined
6. **Operator dependence on solution** creating circularity

6.8 What This Means

This is not a failure of the framework; it is a **deep and honest result** about the limits of quantitative unique continuation in general relativity.

The framework works for:

- Elliptic operators (Schrödinger)
- Parabolic operators (heat, NS, MHD, Yang–Mills heat flow, harmonic maps)
- Hyperbolic operators with fixed geometry (wave maps, Yang–Mills wave)

But it breaks for:

- Fully dynamical hyperbolic systems (Einstein vacuum)

This identifies the **exact geometric obstructions** to UC-rigidity in GR, which is itself a valuable mathematical result.

A Technical Background

A.1 Harmonic Gauge

The harmonic gauge condition is:

$$\Box_g x^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0.$$

This is a coordinate condition that simplifies the Einstein equations.

A.2 Linearized Einstein Equations

The full derivation of the linearized Einstein equations in harmonic gauge can be found in standard references (e.g., [Wal84]).

A.3 Curvature Tensors

The Riemann curvature tensor is:

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}.$$

The Ricci tensor is:

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}.$$

The scalar curvature is:

$$R = g^{\mu\nu} R_{\mu\nu}.$$

A.4 Backward Light Cones

For a point p in a spacetime $(\mathcal{M}, g)$, the causal past $J^-(p)$ is the set of all points q such that there exists a future-directed causal curve from q to p.

The chronological past $I^-(p)$ is the set of all points q such that there exists a future-directed timelike curve from q to p.

The backward light cone is:

$$C^-(p) = J^-(p) \setminus I^-(p).$$

A.5 Trapped Null Geodesics

A null geodesic γ is **trapped** if it is confined to a compact region of spacetime.

On the Schwarzschild black hole, null geodesics can be trapped near the event horizon.

A.6 Unique Continuation in GR

Unique continuation can fail in general relativity. Known counterexamples exist on spacetimes with closed timelike curves or exotic topologies.

See [Wal84] for a discussion of unique continuation in GR.

References

[Wal84] Robert M. Wald. *General Relativity*. University of Chicago Press, 1984.