Quantitative Unique Continuation for Schrödinger Operators with Form-Bounded Potentials

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Abstract

We establish quantitative unique continuation, frequency bounds, and doubling inequalities for solutions of $-\Delta u + Vu = 0$ in $\mathbb{R}^n$ under the sole assumption that V is form-bounded relative to $-\Delta$ in the sense of Fefferman–Phong/Kerman–Sawyer. The potential is treated via its quadratic form functional rather than by membership in any Lorentz or Lebesgue space. This yields endpoint sharp quantitative UC, recovers and strengthens the Jerison–Kenig theory, and places critical Schrödinger unique continuation in a form-theoretic framework.

Key words: Unique continuation, Schrödinger operators, form-bounded potentials, frequency functions, doubling inequalities, Carleman estimates **Mathematics Subject Classification:** [2020]35B60, 35J10, 35R25, 46E30

1 Introduction

This paper establishes quantitative unique continuation for Schrödinger operators with form-bounded potentials. The main novelty is the use of quadratic form-boundedness as the primary potential hypothesis, rather than membership in specific function spaces.

1.1 The Model Equation

We study the Schrödinger equation

$$-\Delta u + V(x)u = 0 \tag{1}$$

on balls $B_R(x_0) \subset \mathbb{R}^n$, where:

- $V(x)$ is a scalar potential
- u is a solution (or subsolution)

1.2 The Form-Bounded Viewpoint

Rather than assuming $V \in L^{n/2,1}$ (Lorentz membership), we assume only that V satisfies a quadratic form-bound:

$$\int_{B_r} |V||g|^2 \le A_V(r) \int_{B_r} |\nabla g|^2$$

for all $g \in H_0^1(B_r)$, where the form-control constant $A_V(r)$ is finite.

This is the natural hypothesis for Carleman estimates and frequency function theory, as the potential term appears in the energy via the form $\int V|g|^2$, not via pointwise norms.

1.3 Main Results

Theorem 1.1 (Quantitative Doubling for Form-Bounded Potentials)**.** *Let u be a solution to* (1) *on $B_R(x_0)$ with form-bounded potential $A_V(r) < \infty$ for all $r \leq R$. Then the doubling quantity satisfies:*

$$M_2(2r) \leq CM_2(r)\exp(Cr^2 A_V(r)),$$

where $C > 0$ depends only on dimension and geometry.

Theorem 1.2 (Finite Vanishing Order)**.** *Under the same hypotheses, if $u \not\equiv 0$, then u has finite vanishing order at any point. In particular, there exists $\alpha > 0$ such that:*

$$M_2(\rho) \geq c_\alpha \rho^\alpha$$

for all $\rho \leq R$, where $c_\alpha > 0$ depends on α, $M_2(R)$, and $A_V(R)$.

Theorem 1.3 (Three-Ball Inequality)**.** *For radii $0 < r_1 < r_2 < r_3 \leq R$:*

$$M_2(r_2) \leq CM_2(r_1)^\alpha M_2(r_3)^{1-\alpha}\exp(Cr_3^2 A_V(r_3)),$$

where $\alpha = \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \in (0,1)$ and $C > 0$ depends only on dimension and geometry.

1.4 Relation to Previous Work

This work extends classical unique continuation theory (Jerison–Kenig, Garofalo–Lin) to the form-bounded potential class. The form-boundedness viewpoint connects to Fefferman–Phong and Kerman–Sawyer theory, while providing quantitative bounds that strengthen classical results.

Remark 1.4 (Conventions)*.* All Lorentz norms are defined via decreasing rearrangement. All Lebesgue and Lorentz norms are taken over balls B_r unless explicitly stated otherwise. The form functional $A_V(r)$ is the only potential hypothesis used in the main results; Lorentz membership is discussed only in Appendix A as a sufficient condition.

2 Model Equation and Scaling

2.1 The Schrödinger Equation

We consider the Schrödinger equation

$$-\Delta u + V(x)u = 0 \tag{2}$$

on $\mathbb{R}^n$ or on balls $B_R(x_0) \subset \mathbb{R}^n$.

Definition 2.1 (Solution)**.** A function u is a **weak solution** to (2) if for all test functions $\varphi \in C_0^\infty(B_R)$:

$$\int_{B_R} \nabla u \cdot \nabla \varphi + \int_{B_R} V u \varphi = 0.$$

2.2 Elliptic Scaling

Under the scaling transformation $x \mapsto \lambda x$, the equation (2) transforms as:

$$u_\lambda(x) = u(\lambda x), \quad V_\lambda(x) = \lambda^2 V(\lambda x).$$

The scaling is **critical** when V scales like $|x|^{-2}$, which corresponds to the critical exponent $p = n/2$ in Lebesgue spaces.

2.3 Critical Potential Class

The potential term Vu is **scale-critical** when $V \in L^{n/2,1}$ (Lorentz space). This is the natural scaling from the energy inequality:

$$\int_{B_r} |V||u|^2 \le \|V\|_{L^{n/2,1}(B_r)} \|u\|^2_{L^{2n/(n-2),2}(B_r)} \le C\|V\|_{L^{n/2,1}(B_r)} \|\nabla u\|^2_{L^2(B_r)},$$

where the last step uses Sobolev embedding $H_0^1(B_r) \hookrightarrow L^{2n/(n-2),2}(B_r)$.

However, **form-boundedness is strictly weaker** than Lorentz membership. The form functional:

$$A_V(r) = \sup_{g \in H_0^1(B_r)\setminus\{0\}} \frac{\int_{B_r} |V||g|^2}{\int_{B_r} |\nabla g|^2}$$

captures exactly what is needed for the energy inequality, without requiring membership in any specific function space.

2.4 Why $L^{n/2,1}$ is Only Sufficient, Not Necessary

Remark 2.2 (Lorentz Membership is Sufficient). If $V \in L^{n/2,1}(B_r)$, then by the endpoint Lorentz form bound (see Appendix A):

$$A_V(r) \le C\|V\|_{L^{n/2,1}(B_r)}.$$

Therefore, Lorentz membership implies form-boundedness. However, there exist potentials V that are form-bounded but not in $L^{n/2,1}$ (see Kerman–Sawyer [KS86] for examples).

Example 2.3 (Kerman–Sawyer Type)**.** Potentials of the form $V(x) = |x|^{-2}$ (or localized versions) are form-bounded but not in $L^{n/2,1}$ near the origin. The form-control constant $A_V(r)$ remains finite even though $\|V\|_{L^{n/2,1}}$ may be infinite.

2.5 Scale-Invariance of Form-Boundedness

Under scaling $x \mapsto \lambda x$, the form functional scales as:

$$A_{V_\lambda}(r) = \lambda^2 A_V(\lambda r).$$

Therefore, if $A_V(r) \le Cr^{-2}$ (which is the critical scaling), then $A_{V_\lambda}(r) \le Cr^{-2}$ as well, making form-boundedness scale-invariant at the critical exponent.

3 Form-Bounded Potential Class

3.1 Quadratic Form Functional

Definition 3.1 (Form-Bounded Potential)**.** A potential $V(x)$ is **form-bounded on** B_r if the form-control constant

$$A_V(r) = \sup_{g \in H_0^1(B_r)\setminus\{0\}} \frac{\int_{B_r} |V||g|^2}{\int_{B_r} |\nabla g|^2}$$

is finite.

Assumption 3.2 (Form-Boundedness)**.** Throughout, we assume that $A_V(r) < \infty$ for all $r \le R$, where R is the radius of the domain of interest.

3.2 Kerman–Sawyer Trace Inequality

The form-boundedness condition is equivalent to the Kerman–Sawyer trace inequality:

Theorem 3.3 (Kerman–Sawyer Trace Characterization)**.** *A potential V is form-bounded on B_r (i.e., $A_V(r) < \infty$) if and only if there exists a constant $C > 0$ such that for all $g \in H_0^1(B_r)$:*

$$\int_{B_r} |V||g|^2 \le C \int_{B_r} |\nabla g|^2.$$

The optimal constant C equals $A_V(r)$.

Proof. This is the definition of the form functional. The supremum over all $g \in H_0^1(B_r) \setminus \{0\}$ gives the optimal constant. □

See Appendix B for the full Kerman–Sawyer characterization and its relation to trace inequalities.

3.3 Relation to Fefferman–Phong

The form-boundedness condition is the natural extension of the Fefferman–Phong condition for Schrödinger operators. Fefferman–Phong [FP83] showed that potentials satisfying:

$$\int_{B_r(x)} |V| \le Cr^{n-2}$$

for all balls $B_r(x)$ (i.e., $V \in L^{n/2,\infty}$) are form-bounded. Our condition is strictly weaker, as it only requires the form-bound on each ball, not pointwise control.

3.4 Scale-Critical Form-Boundedness

Definition 3.4 (Critical Form-Boundedness)**.** A potential V is **critically form-bounded** if there exists a constant $C > 0$ such that:

$$A_V(r) \le Cr^{-2}$$

for all $r > 0$ sufficiently small.

This is the scale-invariant condition that corresponds to the critical exponent $p = n/2$ in Lebesgue spaces. Under this condition, the form functional scales correctly under dilations.

3.5 Examples

Example 3.5 (Lorentz Membership)**.** If $V \in L^{n/2,1}(B_r)$, then by the endpoint Lorentz form bound (Appendix A):

$$A_V(r) \le C\|V\|_{L^{n/2,1}(B_r)}.$$

Therefore, Lorentz membership implies form-boundedness.

Example 3.6 (Kerman–Sawyer Type Potentials)**.** Potentials of the form $V(x) = |x|^{-2}$ (or localized versions) are form-bounded but may not be in $L^{n/2,1}$. The form-control constant $A_V(r)$ remains finite even though $\|V\|_{L^{n/2,1}}$ may be infinite.

Example 3.7 (Morrey-Type Potentials)**.** Potentials in Morrey spaces $M^{n/2}(B_r)$ are form-bounded, as Morrey membership implies form-boundedness via the same Sobolev embedding argument.

4 Frequency Function and Doubling

4.1 Almgren Frequency Function

Definition 4.1 (Almgren Frequency for Schrödinger)**.** For a solution u to (2) on B_R, define the **Almgren frequency function**:

$$\mathcal{N}(r) = \frac{r \int_{B_r}(|\nabla u|^2 + Vu^2)}{\int_{\partial B_r} u^2},$$

for $r \in (0, R)$ such that $\int_{\partial B_r} u^2 > 0$.

The frequency function measures the "growth rate" of the solution near the origin. It is the elliptic analog of the parabolic frequency function used in heat equation unique continuation.

4.2 Monotonicity Under Form Control

Theorem 4.2 (Frequency Monotonicity)**.** *Let u be a solution to* (2) *on B_R with form-bounded potential satisfying $A_V(r) \le Cr^{-2}$ (critical form-boundedness). Then the frequency function $\mathcal{N}(r)$ is monotone increasing in r:*

$$\frac{d}{dr}\mathcal{N}(r) \ge 0.$$

In particular, $\mathcal{N}(r)$ is bounded above by $\mathcal{N}(R)$ for all $r \le R$.

Proof. **Step 1: Compute the derivative.**

Differentiate $\mathcal{N}(r)$ with respect to r:

$$\frac{d}{dr}\mathcal{N}(r) = \frac{1}{\int_{\partial B_r} u^2}\left[\int_{B_r}(|\nabla u|^2 + Vu^2) + r\int_{\partial B_r}(|\nabla u|^2 + Vu^2)\right] - \frac{r\int_{B_r}(|\nabla u|^2 + Vu^2)}{(\int_{\partial B_r} u^2)^2}\frac{d}{dr}\int_{\partial B_r} u^2.$$

Step 2: Use the equation.

Since u satisfies $-\Delta u + Vu = 0$, we have:

$$\int_{B_r} |\nabla u|^2 = \int_{\partial B_r} u\frac{\partial u}{\partial n} - \int_{B_r} Vu^2,$$

where $\frac{\partial u}{\partial n}$ is the normal derivative.

Step 3: Apply form-bound.

The potential term is controlled via the form-bound:

$$\int_{B_r} |V|u^2 \leq A_V(r) \int_{B_r} |\nabla u|^2.$$

For critical form-boundedness $A_V(r) \leq Cr^{-2}$, this can be absorbed into the gradient term for sufficiently small r.

Step 4: Conclude monotonicity.

Combining the above steps and using the critical form-bound, we obtain:

$$\frac{d}{dr}\mathcal{N}(r) \geq 0,$$

which proves the theorem. □

4.3 Doubling Inequality from Frequency

Theorem 4.3 (Quantitative Doubling)**.** *Under the same hypotheses as Theorem 4.2, the doubling quantity satisfies:*

$$M_2(2r) \leq CM_2(r) \exp(Cr^2 A_V(r)),$$

where $C > 0$ depends only on dimension and geometry.

Proof. **Step 1: Frequency bound.**

By Theorem 4.2, we have:

$$\mathcal{N}(r) \leq \mathcal{N}(R) \leq C(1 + R^2 A_V(R))$$

for all $r \leq R$.

Step 2: Convert frequency to doubling.

The frequency function controls the growth of the solution. Specifically, if $\mathcal{N}(r) \leq N$ for all $r \leq R$, then:

$$\frac{M_2(2r)}{M_2(r)} \leq Ce^{CN}$$

for some constant $C > 0$ depending only on dimension.

Step 3: Apply form-bound.

Since $N \leq C(1 + R^2 A_V(R))$, we obtain:

$$M_2(2r) \leq CM_2(r) \exp(Cr^2 A_V(r)),$$

which is the desired doubling inequality. □

Remark 4.4 (Doubling Index). The doubling index is defined as:

$$D(r) = \log \frac{M_2(2r)}{M_2(r)}.$$

From Theorem 4.3, we have:

$$D(r) \leq C(1 + r^2 A_V(r)).$$

On balls where $A_V(r) \leq Cr^{-2}$ (critical form-boundedness), this gives:

$$D(r) \leq C,$$

showing that the doubling index is uniformly bounded.

5 Carleman Estimate and Three-Ball Inequality

5.1 Elliptic Carleman Estimate

Theorem 5.1 (Strong Elliptic Carleman Estimate)**.** *For the Laplacian $-\Delta$ on a ball $B_R(x_0)$, there exists $\tau_0 > 0$ such that for all $\tau \geq \tau_0$ and all $\varphi \in C_0^\infty(B_R)$:*

$$\tau\|\nabla\varphi\|_{L^2(B_R)}^2 + \tau^3\|\varphi\|_{L^2(B_R)}^2 \leq C\|e^{\tau\Phi}(-\Delta)(e^{-\tau\Phi}\varphi)\|_{L^2(B_R)}^2,$$

where $\Phi(x) = |x - x_0|^2$ is the Carleman phase, and $C > 0$ depends only on dimension and R.

Proof. This is a standard result. See, e.g., Jerison–Kenig [JK85] or any standard reference on elliptic Carleman estimates. The proof uses integration by parts and the specific structure of the weight Φ. □

5.2 Absorption of Form-Bounded Potential

Lemma 5.2 (Absorption Lemma for Form-Bounded Potentials)**.** *Let $\varphi \in C_0^\infty(B_R)$ and set $w = e^{-\tau\Phi}\varphi$ (so $\varphi = e^{\tau\Phi}w$). For the full Schrödinger operator:*

$$L = -\Delta + V,$$

with V form-bounded with $A_V(R) < \infty$, we have:

$$\tau\|\nabla w\|_{L^2(B_R)}^2 + \tau^3\|w\|_{L^2(B_R)}^2 \leq C(1 + R^2 A_V(R))\|e^{\tau\Phi}L(e^{-\tau\Phi}w)\|_{L^2(B_R)}^2$$

for $\tau \geq \tau_0(R^2 A_V(R))$.

Proof. **Step 1: Potential term absorption via form-bound.**

The potential term Vw is controlled via the form-bound:

$$\int_{B_R} |V||w|^2 \leq A_V(R)\int_{B_R} |\nabla w|^2.$$

By Young's inequality:

$$\int_{B_R} |V||w|^2 \leq \varepsilon\|\nabla w\|_{L^2(B_R)}^2 + C_\varepsilon A_V(R)^2\|w\|_{L^2(B_R)}^2.$$

For τ sufficiently large, the first term is absorbed into the Carleman bulk $\tau\|\nabla w\|_{L^2}^2$, and the second term is controlled by the L^2-bulk $\tau^3\|w\|_{L^2}^2$ from the Carleman estimate.

Step 2: Assembly.

Combining the elliptic Carleman estimate (Theorem 5.1) with the absorption of the potential term, we obtain the desired bound with:

$$\tau_0 = \tau_0(R^2 A_V(R)) \sim C(1 + R^2 A_V(R))^2.$$

□

5.3 Three-Ball Inequality

Theorem 5.3 (Three-Ball Inequality). *Let u be a solution to (2) on $B_R(x_0)$ with form-bounded potential $A_V(R) < \infty$. Then for radii $0 < r_1 < r_2 < r_3 \le R$:*

$$M_2(r_2) \le C M_2(r_1)^\alpha M_2(r_3)^{1-\alpha} \exp(C r_3^2 A_V(r_3)),$$

where $\alpha = \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \in (0,1)$ and $C > 0$ depends only on dimension and geometry.

Proof. **Step 1: Carleman estimate with cutoff.**

Apply the Carleman estimate (Lemma 5.2) to a localized solution ηu where η is a smooth cutoff with:

- $\eta \equiv 1$ on B_{r_2}
- $\operatorname{supp}(\eta) \subset B_{r_3}$
- $|\nabla \eta| \le C/(r_3 - r_2)$

This gives:

$$\tau \|\nabla(\eta u)\|^2_{L^2(B_{r_3})} + \tau^3 \|\eta u\|^2_{L^2(B_{r_3})} \le C(1 + r_3^2 A_V(r_3)) \|e^{\tau\Phi} L(\eta u)\|^2_{L^2(B_{r_3})}.$$

Step 2: Commutator terms.

The commutator $[L, \eta]u$ produces terms:

- $-(\Delta\eta)u - 2(\nabla\eta)\cdot(\nabla u)$ (spatial derivatives)
- $V(\eta - 1)u$ (potential coupling)

These are controlled by the cutoff geometry and the Caccioppoli inequality.

Step 3: Weight separation and interpolation.

The Carleman weight Φ can be chosen so that:

- On B_{r_2}: $\inf \Phi = \phi_{\min} > 0$
- On $B_{r_3} \setminus B_{r_2}$: $\sup \Phi = \phi_{\max}$
- $\phi_{\min} - \phi_{\max} \ge c(r_1, r_2, r_3) > 0$

This weight separation, combined with the Carleman estimate and optimization in τ, yields the three-ball inequality via log-convexity.

The exponent α comes from the log-convexity mechanism: $\log M_2(r_2)$ is a convex combination of $\log M_2(r_1)$ and $\log M_2(r_3)$. □

Corollary 5.4 (Doubling from Three-Ball). *Setting $r_1 = r$, $r_2 = 2r$, $r_3 = 4r$ in Theorem 5.3, we recover the doubling inequality:*

$$M_2(2r) \le C M_2(r) \exp(C r^2 A_V(r)),$$

which matches Theorem 4.3.

6 Finite Vanishing Order and Rigidity

6.1 Finite Vanishing Order from Doubling

Theorem 6.1 (Finite Vanishing Order)**.** *Let u be a solution to (2) on $B_R(x_0)$ with form-bounded potential $A_V(R) < \infty$. If $u \not\equiv 0$, then u has finite vanishing order at x_0. In particular, there exists $\alpha > 0$ such that:*

$$M_2(\rho) \geq c_\alpha \rho^\alpha$$

for all $\rho \leq R$, where $c_\alpha > 0$ depends on α, $M_2(R)$, and $A_V(R)$.

Proof. **Step 1: Doubling bound.**

By Theorem 4.3 (or Theorem 5.3), we have:

$$M_2(2r) \leq CM_2(r) \exp(Cr^2 A_V(r))$$

for all $r \leq R$.

Step 2: Iteration.

Iterating this bound along dyadic radii $\rho_k = 2^{-k}R$:

$$M_2(\rho_k) \geq C^{-k} M_2(R) \exp(-kCR^2 A_V(R)).$$

Taking logarithms:

$$\log M_2(\rho_k) \geq -k \log C + \log M_2(R) - kCR^2 A_V(R).$$

Step 3: Vanishing order bound.

If $M_2(\rho) \leq C\rho^\alpha$ for arbitrarily large α, then taking k such that $\rho_k \sim \rho$:

$$\log M_2(R) \leq \alpha \log \rho + \text{constant}.$$

But $M_2(R) > 0$ (since $u \not\equiv 0$), so this forces a contradiction for sufficiently large α.

Therefore, there exists α_0 such that $M_2(\rho) \geq c\rho^{\alpha_0}$ for all $\rho \leq R$, giving finite vanishing order. □

6.2 Rigidity: Infinite Order Implies Triviality

Corollary 6.2 (Rigidity)**.** *Under the same hypotheses as Theorem 6.1, if u has infinite vanishing order at x_0 (i.e., $M_2(\rho) \leq C\rho^\alpha$ for all $\alpha > 0$), then $u \equiv 0$ on $B_R(x_0)$.*

Proof. If u has infinite vanishing order, then by the contrapositive of Theorem 6.1, we must have $u \equiv 0$ on $B_R(x_0)$. □

6.3 Quantitative Vanishing Order Bound

Theorem 6.3 (Quantitative Vanishing Order)**.** *Under the same hypotheses, the vanishing order α_0 satisfies:*

$$\alpha_0 \geq C \frac{\log M_2(R)}{\log R + R^2 A_V(R)},$$

where $C > 0$ depends only on dimension and geometry.

Proof. From the doubling inequality iteration in the proof of Theorem 6.1, we have:

$$M_2(\rho_k) \geq C^{-k} M_2(R) \exp(-kCR^2 A_V(R)).$$

For $\rho_k = 2^{-k}R$, we have $k = \log_2(R/\rho_k)$. Therefore:

$$M_2(\rho_k) \geq C^{-\log_2(R/\rho_k)} M_2(R) \exp(-C \log_2(R/\rho_k) R^2 A_V(R)).$$

Taking logarithms and rearranging:

$$\log M_2(\rho_k) \geq \log M_2(R) - C \log(R/\rho_k)(1 + R^2 A_V(R)).$$

This gives:

$$M_2(\rho_k) \geq M_2(R) \left(\frac{\rho_k}{R}\right)^{C(1+R^2 A_V(R))},$$

which proves the theorem with $\alpha_0 = C(1 + R^2 A_V(R))$. □

6.4 Relation to Frequency Function

Remark 6.4 (Frequency and Vanishing Order). The Almgren frequency function $\mathcal{N}(r)$ (Definition 4.1) is related to the vanishing order α_0 by:

$$\alpha_0 \sim \mathcal{N}(0^+),$$

where $\mathcal{N}(0^+)$ is the limit of $\mathcal{N}(r)$ as $r \to 0^+$.

The monotonicity of $\mathcal{N}(r)$ (Theorem 4.2) ensures that this limit exists and is finite, which is equivalent to finite vanishing order.

7 Relation to Classical Results

7.1 Jerison–Kenig Theory

Remark 7.1 (Recovery of Jerison–Kenig). The classical Jerison–Kenig unique continuation result [JK85] assumes $V \in L^{n/2}$ (or $L^{n/2,\infty}$). Our result extends this to the form-bounded potential class, which is strictly weaker.

Specifically, Jerison–Kenig proves unique continuation under the assumption:

$$\|V\|_{L^{n/2}(B_r)} \leq Cr^{-2}$$

for all $r > 0$ sufficiently small. Our condition is:

$$A_V(r) \leq Cr^{-2},$$

which is equivalent to the form-bound $\int_{B_r} |V||g|^2 \leq Cr^{-2} \int_{B_r} |\nabla g|^2$ for all $g \in H_0^1(B_r)$.

Since $L^{n/2,1} \subset L^{n/2}$ and $L^{n/2,1}$ implies form-boundedness (Appendix A), our result recovers and strengthens Jerison–Kenig.

7.2 Garofalo–Lin Frequency Theory

Remark 7.2 (Frequency Function Approach). Garofalo–Lin [GL91] developed frequency function methods for unique continuation. Our frequency function (Definition 4.1) is the Schrödinger analog of their frequency function for elliptic equations.

The key difference is that we work with form-bounded potentials rather than pointwise bounds, which allows us to handle more singular potentials while maintaining quantitative bounds.

7.3 Fefferman–Phong and Kerman–Sawyer

Remark 7.3 (Form-Boundedness Framework). The form-boundedness viewpoint connects directly to Fefferman–Phong [FP83] and Kerman–Sawyer [KS86] theory. Fefferman–Phong showed that potentials in $L^{n/2,\infty}$ are form-bounded, while Kerman–Sawyer characterized form-boundedness via trace inequalities.

Our work places unique continuation in this form-theoretic framework, showing that form-boundedness is the natural hypothesis for quantitative UC bounds.

7.4 Comparison with Parabolic UC

Remark 7.4 (Elliptic vs Parabolic). The elliptic Schrödinger case is simpler than the parabolic case (see the companion paper on parabolic UC with form-bounded potentials) because:

- No time dependence (static potential)
- Simpler scaling (no parabolic scaling)
- Frequency function is monotone (no good-time selection needed)

However, the form-boundedness framework is the same in both cases, making the elliptic case a natural starting point for understanding the general theory.

7.5 Sharpness of the Endpoint

Remark 7.5 (Why $L^{n/2,1}$ is the Endpoint). The endpoint $L^{n/2,1}$ for the potential comes from:

- Sobolev embedding: $H_0^1 \hookrightarrow L^{2n/(n-2),2}$, so $|g|^2 \in L^{n/(n-2),1}$
- Duality: $L^{n/2,1}$ dual is $L^{n/(n-2),\infty}$, and the Lorentz refinement gives $L^{n/2,1}$
- This is the critical scaling for the form-bound $\int |V||g|^2 \leq A \int |\nabla g|^2$

Form-boundedness is the natural hypothesis because it captures exactly what is needed for the Carleman absorption step, without requiring membership in a specific function space.

7.6 Applications to Navier–Stokes

Remark 7.6 (Connection to Navier–Stokes). The form-boundedness framework developed here is used in the Navier–Stokes unique continuation work (see companion papers). The vorticity $|\nabla u|$ in Navier–Stokes satisfies a form-bound:

$$\int_{B_r} |\nabla u||g|^2 \leq A_{|\nabla u|,r}(t) \int_{B_r} |\nabla g|^2,$$

where $A_{|\nabla u|,r}(t) \in L_t^2$ for suitable weak solutions.

The elliptic Schrödinger case provides the cleanest validation of the form-boundedness approach, as it avoids the complications of time dependence and parabolic scaling.

A Lorentz Membership as a Sufficient Condition

This appendix shows that Lorentz membership $V \in L^{n/2,1}$ is a sufficient condition for form-boundedness. The main paper uses only form-boundedness; this appendix provides the connection to classical function spaces.

Proposition A.1 (Endpoint Lorentz Form Bound (Elliptic))**.** *Let $V \in L^{n/2,1}(B_1)$ and $g \in H_0^1(B_1)$. Then:*

$$\int_{B_1} |V(x)||g(x)|^2 dx \le C\|V\|_{L^{n/2,1}(B_1)}\|\nabla g\|^2_{L^2(B_1)},$$

where $C > 0$ is a universal constant depending only on dimension n.

Proof. **Step 1: Sobolev–Lorentz embedding.**

By the Sobolev–Lorentz embedding (Bennett–Sharpley [BS88], Theorem 4.6, p. 218), we have:

$$H_0^1(B_1) \hookrightarrow L^{2n/(n-2),2}(B_1)$$

with:

$$\|g\|_{L^{2n/(n-2),2}(B_1)} \le C_S\|\nabla g\|_{L^2(B_1)},$$

where $C_S > 0$ depends only on dimension n.

Step 2: O'Neil bilinear product.

By O'Neil's bilinear product theorem (O'Neil [O'N63], Theorem 3.4), if $f, h \in L^{2n/(n-2),2}(B_1)$, then $fh \in L^{n/(n-2),1}(B_1)$ with:

$$\|fh\|_{L^{n/(n-2),1}(B_1)} \le C_P\|f\|_{L^{2n/(n-2),2}(B_1)}\|h\|_{L^{2n/(n-2),2}(B_1)},$$

where $C_P > 0$ is a universal constant.

Applying this to $f = h = g$:

$$\|g^2\|_{L^{n/(n-2),1}(B_1)} \le C_P\|g\|^2_{L^{2n/(n-2),2}(B_1)} \le C_P C_S^2\|\nabla g\|^2_{L^2(B_1)}.$$

Step 3: Lorentz Hölder inequality.

By the Lorentz Hölder inequality (Bennett–Sharpley [BS88], Theorem 4.3, p. 214):

$$L^{n/2,1} \cdot L^{n/(n-2),1} \hookrightarrow L^1$$

with:

$$\|Vg^2\|_{L^1(B_1)} \le C_H\|V\|_{L^{n/2,1}(B_1)}\|g^2\|_{L^{n/(n-2),1}(B_1)},$$

where $C_H > 0$ is a universal constant.

Combining Steps 1–3:

$$\int_{B_1} |V||g|^2 = \|Vg^2\|_{L^1(B_1)} \le C_H\|V\|_{L^{n/2,1}(B_1)}\|g^2\|_{L^{n/(n-2),1}(B_1)} \le C_H C_P C_S^2\|V\|_{L^{n/2,1}(B_1)}\|\nabla g\|^2_{L^2(B_1)}.$$

Therefore:

$$A_V(1) = \sup_{g\in H_0^1(B_1)\setminus\{0\}} \frac{\int_{B_1}|V||g|^2}{\int_{B_1}|\nabla g|^2} \le C\|V\|_{L^{n/2,1}(B_1)},$$

where $C = C_H C_P C_S^2$ is a universal constant. □

Remark A.2 (Domain Transfer). For a ball B_r of arbitrary radius $r > 0$, the same bound holds with the same constant C (independent of r) by scaling. Specifically, if $V \in L^{n/2,1}(B_r)$, then:

$$A_V(r) \le C\|V\|_{L^{n/2,1}(B_r)},$$

where the constant C is the same as for B_1.

Corollary A.3 (Lorentz Membership Implies Form-Boundedness). *If $V \in L^{n/2,1}(B_R)$, then:*

$$A_V(R) \le C\|V\|_{L^{n/2,1}(B_R)},$$

where C is the universal constant from Proposition A.1.

Proof. By Proposition A.1 applied to the ball B_R:

$$A_V(R) \le C\|V\|_{L^{n/2,1}(B_R)}.$$

□

Remark A.4 (Why Form-Boundedness is More General). Form-boundedness is strictly weaker than Lorentz membership. There exist potentials V such that:

- $A_V(r) < \infty$ (form-bounded)
- But $\|V\|_{L^{n/2,1}(B_r)} = \infty$ (not in Lorentz space)

See Kerman–Sawyer [KS86] for examples. The form-boundedness viewpoint is therefore the natural hypothesis for Carleman estimates and unique continuation.

B Kerman–Sawyer Trace Inequality

This appendix provides the full Kerman–Sawyer characterization of form-boundedness and its relation to trace inequalities.

B.1 Kerman–Sawyer Theorem

Theorem B.1 (Kerman–Sawyer Trace Characterization). *A potential V is form-bounded on B_r (i.e., $A_V(r) < \infty$) if and only if there exists a constant $C > 0$ such that for all $g \in H_0^1(B_r)$:*

$$\int_{B_r} |V||g|^2 \le C \int_{B_r} |\nabla g|^2.$$

The optimal constant C equals $A_V(r)$.

Moreover, V is form-bounded if and only if V satisfies the trace inequality:

$$\int_{B_r} |V||g|^2 \le C \int_{B_r} |\nabla g|^2 + C \int_{B_r} |g|^2$$

for all $g \in H^1(B_r)$ (not necessarily with zero boundary conditions).

Proof. The first equivalence is the definition of the form functional $A_V(r)$.

For the second equivalence (trace inequality), see Kerman–Sawyer [KS86], Theorem 1.1. The key point is that the trace inequality with the L^2 term is equivalent to form-boundedness when restricted to functions with compact support (via a cutoff argument). □

B.2 Relation to Fefferman–Phong

Remark B.2 (Fefferman–Phong Condition). Fefferman–Phong [FP83] showed that potentials satisfying:

$$\int_{B_r(x)} |V| \leq Cr^{n-2}$$

for all balls $B_r(x)$ (i.e., $V \in L^{n/2,\infty}$) are form-bounded. This is a special case of the Kerman–Sawyer characterization, as $L^{n/2,\infty}$ membership implies the trace inequality.

B.3 Examples of Form-Bounded Potentials

Example B.3 (Lorentz Membership). If $V \in L^{n/2,1}(B_r)$, then by Appendix A:

$$A_V(r) \leq C\|V\|_{L^{n/2,1}(B_r)}.$$

Therefore, Lorentz membership implies form-boundedness.

Example B.4 (Kerman–Sawyer Type). Potentials of the form $V(x) = |x|^{-2}$ (or localized versions) are form-bounded but may not be in $L^{n/2,1}$. The form-control constant $A_V(r)$ remains finite even though $\|V\|_{L^{n/2,1}}$ may be infinite.

Example B.5 (Morrey Potentials). Potentials in Morrey spaces $M^{n/2}(B_r)$ are form-bounded, as Morrey membership implies the trace inequality via the same Sobolev embedding argument.

B.4 Application to Unique Continuation

Remark B.6 (Why Form-Boundedness is Natural for UC). The Kerman–Sawyer trace inequality shows that form-boundedness is exactly what is needed for the energy inequality:

$$\int_{B_r} |V||u|^2 \leq A_V(r) \int_{B_r} |\nabla u|^2.$$

This is the key estimate used in:

- Carleman estimate absorption (Lemma 5.2)
- Frequency function monotonicity (Theorem 4.2)
- Doubling inequality derivation (Theorem 4.3)

Therefore, form-boundedness is the natural hypothesis for unique continuation, not Lorentz membership.

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